Assignment 5 Solution

Exercise 1:

a) Let a + b = 2n + 1
   Let b + c = 2m + 1
   a + 2b + c = 2(n + m) + 2
   a + c = 2(n + m – b + 2)
   ⇨ (a + c) is even [Direct proof]

b) RTP: ab is odd → a is odd and b is odd
   Suppose that a or b are even

   If a is even, then a = 2k, then ab = 2k*b, then ab is even
   If b is even, then b = 2m, then ab = 2m*a, then ab is even
   then:

   a or b is even → ab is even [Direct proof]

   ⇨ ab is odd → a is odd and b is odd [Contraposition]

Exercise 2:

Let a be an irrational number and b, c, d and e integers different from zero

- a irrational
- $\frac{b}{c}$ and $\frac{d}{e}$ rational,
  - which means $-\frac{b}{c}$ is also rational
  - and $\frac{c}{b}$ is also rational

a) RTP: the sum of a rational number and an irrational number is irrational
   Now, Suppose that the sum of irrational number and rational number is rational; then a $+ \frac{\mathrm{b}}{\mathrm{c}} = \frac{\mathrm{d}}{\mathrm{e}}$, for some values of {a, b, c, d, e}; then
   $a = \frac{\mathrm{cd-be}}{\mathrm{ce}}$
   ⇨ but $\frac{\mathrm{cd-be}}{\mathrm{ce}}$ is rational since the addition of two rational numbers is rational
   ⇨ then a is rational (contradiction)
   ⇨ The sum of an irrational and a rational number is irrational

b) RTP: the product of a nonzero rational number and an irrational number is irrational
Suppose that the product of a nonzero rational number and an irrational number is rational
So $a \times \frac{b}{c} = \frac{d}{e}$
$a = \frac{cd}{be}$ which is rational (contradiction)
⇨ The product of a nonzero rational number and an irrational number is irrational

Exercise 3:

- P(1) is if n is a positive integer which in this case it is (1) then 2 ≥ 2 which is true [Constructive Existence Proof]
- 2*n >= n+1 → n >= 1, 1>=1, implies P(1) is true [Direct Proof]

Exercise 4:

RTP: If n is perfect cube, then n+3 is not a perfect cube

Assume that their exists n such that it is a perfect cube, and n+3 is also a perfect cube

Then $n = a^3$, and $n+3 = b^3$, for some integers *a* and *b*
$b^3 - a^3 = (b - a)(b^2 + ba + a^2) = 3$
since a and b are integers, then (b-a) is integer, and $(b^2 + ba + a^2)$, then one of these is correct:

- $(b - a)=3$ and $(b^2 + ba + a^2)=1$ ==>b=3+a ==> $0=8+9a+2a^2$, then a is not integer ==> contradiction
- $(b - a)=1$ and $(b^2 + ba + a^2)=3$==>b=1+a ==> $0=-2+3a+3a^2$, then a is not integer ==> contradiction

Then if n is a perfect cube, n+3 can't be a perfect cube

Exercise 5:

RTP: n is odd ↔ 3n + 2 is odd

RTP: n is odd → 3n + 2 is odd (1)

3n + 2 is odd → n is odd (2)

To prove (1), we use direct proof

n = 2k + 1

3(2k + 1) + 2 = 2(3k + 2) + 1 = 2s + 1 which is odd (proved)

To prove (2), we use contraposition

Suppose that n is even so n = 2m

3(2m) + 2 = 2(3m + 1) = 2t which is even

- ⇨ 3n + 2 is odd → n is odd
- ⇨ n is odd ↔ 3n + 2 is odd

Exercise 6:

No. This line of reasoning shows that if $\sqrt{5x^2 - 4} = x$, then we must have x = 1 or x = − 1. These are therefore the only possible solutions, but we have no guarantee that they are solutions, since not all of our steps were reversible (in particular, squaring both sides). Therefore we must substitute these values back into the original equation to determine whether they do indeed satisfy it.

Exercise 7:

There exists n such that $\sum_{k=0}^{n-1} k = n$

Let n = 3 => 0 + 1 + 2 = 3 [Constructive]

Exercise 8:

1. x is rational → x = $\frac{a}{b}$ where a and b are integers and b is not 0
   a. $\frac{x}{3} = \frac{a}{3b} = \frac{a}{c}$ where c = 3b => $\frac{x}{3}$ is rational [Direct proof]
   b. $5x - 2 = \frac{5a}{b} - 2 = \frac{5a-2b}{b} = \frac{d}{b}$ => 5x – 2 is rational [Direct proof]
2. $\frac{x}{3}$ is rational → $\frac{x}{3} = \frac{a}{b}$
   a. $\frac{x}{3} = \frac{a}{b}$ → $x = \frac{3a}{c}$ => x is rational [Direct proof]
   b. x is rational (a), then 5x is rational, then 5x – 2 is rational [Direct proof]
3. 5x – 2 is rational → $5x - 2 = \frac{a}{b}$
   a. → $5x = \frac{a}{b} - 2$ → $x = \frac{a-2b}{5b}$ then x is rational [Direct proof]
   b. then $\frac{x}{3}$ is rational from (1) [Direct proof]

x is rational ↔ x/3 is rational ↔ 5x+2 is rational

Exercise 9:

1. n^2 is even then n is even as shown in Ex 1-b, since if n is odd, n^2 is odd ==> n =2k
   a. 1-n = 1-2k = -(2k-1), then 1-n is odd
   b. n^2 is even and n is even, then n^3 as shown in Ex 1-b
   c. n^2+1, then $n^2+1 = (2k)^2+1$, then $n^2+1 = 2(2k^2)+1$, then n^2+1 is odd

2. 1-n is odd, then 1-n=2k+1
   a. 1-n =2k+1, then n = -2k, then n is even, then n^2 is even as shown in Ex 1-b
   b. n is even and n^2 is even, then n^3 is even as shown in Ex 1-b
   c. n = -2k, then $n^2+1 = 4k + 1 = 2(2k) +1$ then n^2+1 is odd

3. n^3 is even, then n^2 or n is even, as shown in Ex1-b;
   a. if n^2 is even, then n is even.
   b. If n is even then n^2 is even as shown in Ex1-b
   - ⇨ n^3 → n and n^2 are even, then all the other statements follow as shown in part 1

4. if n^2+1 is odd, then $n^2+1 = 2k+1$ then $n^2 = 2k$ then n^2 is even, then all the other statements follow as shown in part 1

n^2 is even ↔ 1-n is odd ↔ n^3 is even ↔ n^2+1 is odd